# The shape theorem for the frog model*

O.S.M. Alves F.P. Machado S.Yu. Popov

February 1, 2008

Department of Statistics, Institute of Mathematics and Statistics, University of São Paulo, Rua do Matão 1010, CEP 05508–900, São Paulo SP, Brazil
e-mails: oswaldo@ime.usp.br, fmachado@ime.usp.br, popov@ime.usp.br

**Abstract**

In this work we prove a shape theorem for a growing set of Simple Random Walks (SRWs) on $\mathbb{Z}^d$, known as frog model. The dynamics of this process is described as follows: There are active particles, which perform independent SRWs, and sleeping particles, which do not move. When a sleeping particle is hit by an active particle, it becomes active too. At time 0 all particles are sleeping, except for that placed at the origin. We prove that the set of the original positions of all the active particles, rescaled by the elapsed time, converges to some compact convex set. In some specific cases we are able to identify (at least partially) this set.

**Keywords:** frog model, shape theorem, subadditive ergodic theorem

## 1 Introduction and results

We study a discrete time particle system on $\mathbb{Z}^d$ named frog model. In this model particles, thought of as frogs, move as independent simple random walks (SRWs) on $\mathbb{Z}^d$. At time zero there is one particle at each site of the lattice and all the particles are sleeping except for the one at the origin. The only awakened particle starts to perform a SRW. From then on when an

*The authors are thankful to CAPES/PICD, CNPq (300226/97–7) and FAPESP (97/12826–6) for financial support.

awakened particle jumps on the top of a sleeping particle, the latter wakes up and starts jumping independently, also performing a SRW. The number of awakened particles grows to infinity as active particles jump on sites that have not been visited before, awakening the particles that are sitting there. Let us underline that the active particles do not interact with each other and there is no “one-particle-per-site” rule.

This model is a modification of a model for information spreading that the authors have learned from K. Ravishankar. The idea is that every moving particle has some information and it shares that information with a sleeping particle at the time the former jumps on the back of the latter. Particles that have the information move freely helping in the process of spreading information. The model that we deal with in this paper is a discrete-time version of that proposed by R. Durrett, who also suggested the term “frog model”.

In [6] this model was studied from the point of view of transience and recurrence in the case when the initial configuration of sleeping particles is random and has decaying density.

We now define the process in a more formal way. Let $\{(S_n^x)_{n\in\mathbb{N}}, x \in \mathbb{Z}^d\}$ be independent SRWs such that $S_0^x = x$ for all $x \in \mathbb{Z}^d$. For the sake of cleanness let $S_n := S_n^0$. These sequences of random variables give the trajectory of the particle seated originally at site $x$, starting to move at the time it wakes up. Let

$$t(x,z) = \min\{n : S_n^x = z\}, \tag{1.1}$$

remembering that if $d > 2$ then $\mathbf{P}[t(x,z) = \infty] > 0$.

We define now

$$T(x,z) = \inf\Big\{\sum_{i=1}^{k} t(x_{i-1},x_i) : x_0 = x, \dots, x_k = z \text{ for some } k\Big\}, \tag{1.2}$$

the passage time from $x$ to $z$ for the frog model. By now it should be clear that if the process starts from just one active particle sitting at site $x$, in the sense that that particle is the only active one, $T(x,z)$ is the time it takes to have the particle sitting at site $z$ to be awakened. Note that the particle which awakens $z$ need not be that from $x$.

Now, let $Z_y^x(n)$ be the location (at time $n$) of the particle that started from site $y$ in the process in which the only active particle at time zero was

at site $x$. Formally, we have $Z^x_x(n) = S^x_n$, and

$$Z^x_y(n) = \begin{cases} y, & \text{if } T(x,y) \geq n, \\ S^y_{n-T(x,y)}, & \text{if } T(x,y) < n. \end{cases}$$

Since every random variable of the form $Z^x_y(n)$ is constructed using the random variables $\{(S^x_n)_{n\in\mathbb{N}}, x \in \mathbb{Z}^d\}$ and the related random variables $\{T(x,y) : x, y \in \mathbb{Z}^d\}$, this defines a coupling of processes $\{Z^x, x \in \mathbb{Z}^d\}$. We point out that a particle starting from a given site, as soon as it wakes up (becomes active), executes the same random walk in all the processes. The only difference for that particle is in the time when it wakes up.

With the help of these variables we define the sites whose originally sleeping particles have been awakened by time $n$, by some originally awakened particle from the set $\mathsf{B}$. Namely

$$\xi^{\mathsf{B}}_n = \{y \in \mathbb{Z}^d : T(x,y) \leq n \text{ for some } x \in \mathsf{B}\}.$$

We are mostly concerned with $\xi_n := \xi^0_n$ (note that the process $\xi_n$ itself is *not* Markovian). Moreover, we define

$$\bar{\xi}_n = \{x + (-1/2, 1/2]^d : x \in \xi_n\} \subset \mathbb{R}^d.$$

It is a basic fact that the displacement of a single SRW at time $n$ is roughly $\sqrt{n}$. Here we prove that in the frog model the particles may "help" each other in order to make the boundary of $\bar{\xi}_n$ grow linearly with time. Namely, the main result of this paper is the following

**Theorem 1.1** *There is a non-empty convex set* $\mathsf{A} \subset \mathbb{R}^d$ *such that, for any* $0 < \varepsilon < 1$

$$(1-\varepsilon)\mathsf{A} \subset \frac{\bar{\xi}_n}{n} \subset (1+\varepsilon)\mathsf{A}$$

*for all* $n$ *large enough almost surely.*

Note that, although Theorem 1.1 establishes the existence of the asymptotic shape $\mathsf{A}$, it is difficult to say something definite about this shape, except, of course, that $\mathsf{A}$ is symmetric and that $\mathsf{A} \subset \mathcal{D}$, where

$$\mathcal{D} = \{x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d : |x^{(1)}| + \cdots + |x^{(d)}| \leq 1\}.$$

Also, note that if the initial configuration is augmented (i.e. some new particles are added), then the asymptotic shape (when it exists) augments as well. We are going to show that if the initial configuration is rich enough, then the limiting shape $\mathsf{A}$ may contain some pieces of the boundary of $\mathcal{D}$ (a "flat edge" result) or even coincide with $\mathcal{D}$ (a "full diamond" result).

To formulate these results, we need some additional notation. For $1 \leq i < j \leq d$ let

$$\Lambda_{ij} = \{x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d : x^{(k)} = 0 \text{ for } k \neq i, j\},$$

and for $0 < \beta < 1/2$ let

$$\Theta_{ij}^{\beta} = \{x = (x^{(1)}, \dots, x^{(d)}) \in \Lambda_{ij} : |x^{(i)}| + |x^{(j)}| = 1, \min\{|x^{(i)}|, |x^{(j)}|\} \geq \beta\}.$$

Define $\Theta^{\beta}$ to be the convex hull of $(\Theta_{ij}^{\beta})_{1 \leq i < j \leq d}$. Denote by $\mathsf{A}_m$ the asymptotic shape in the frog model when the initial configuration is such that any site $x \in \mathbb{Z}^d$ contains exactly $m$ particles. The existence of $\mathsf{A}_m$ for arbitrary $m$ can be derived in just the same way as in the case $m = 1$ (Theorem 1.1). Then, for a positive integer-valued random variable $\eta$ denote by $\mathsf{A}_\eta$ the asymptotic shape (if exists) for the frog model with the initial configuration constructed in the following way: Into every site we put a random number of particles independently of other sites, and the distribution of this random number is that of $\eta$.

**Theorem 1.2** *If $m$ is large enough, then there exists $0 < \beta < 1/2$ such that $\Theta^{\beta} \subset \mathsf{A}_m$.*

**Theorem 1.3** *Suppose that for some positive $\delta < d$ and for all $n$ large enough we have*

$$\mathbf{P}[\eta \geq (2d)^n] \geq n^{-\delta}, \tag{1.3}$$

*and $\eta \geq 1$ a.s. Then $\mathsf{A}_\eta = \mathcal{D}$.*

The paper is organized in the following way. Section 2 contains some well known results about large deviations and SRW on $\mathbb{Z}^d$. We need these results to verify the hypotheses of Liggett's subadditive ergodic theorem. These hypotheses are verified in Section 3 and the proof of the shape theorem is given in Section 4. Besides that, in Section 4 we prove the "flat edge" and the "full diamond" results.

# 2 Basic facts

Along this section we state basic facts about large deviations and random walks which we need to prove our results. A couple of them are followed by their proofs just because we have not been able to find them in the bibliography. As usual, $C, C_1, C_2, \ldots$ stand for positive finite constants. For what follows we use these constants freely. Also, $\lfloor x \rfloor$ stands for the largest integer which is less than or equal to $x$, while $\lceil x \rceil$ is the smallest integer which is greater than or equal to $x$.

**Lemma 2.1** *If $X$ is a random variable assuming positive integer values such that $X \leq a$ almost surely and $\mathbf{E}X \geq b > 0$ then*

$$\mathbf{P}\Big[X \geq \frac{b}{2}\Big] \geq \frac{b}{2a}. \tag{2.1}$$

*Proof.* We have

$$\begin{aligned} b \leq \mathbf{E}X &= \sum_{i<b/2} i\mathbf{P}[X=i] + \sum_{i \geq b/2} i\mathbf{P}[X=i] \\ &\leq \frac{b}{2} + a\mathbf{P}\Big[X \geq \frac{b}{2}\Big], \end{aligned}$$

and (2.1) follows. □

## 2.1 On large deviations

In this subsection we state two large deviations results. The first one (Lemma 2.2) is a simple application of the exponential Chebyshev inequality to sums of independent Bernoulli random variables. The second one (Lemma 2.3) is useful when one has positive integer random variables but cannot guarantee the existence of their moment generating functions. That condition is weakened and substituted by a sub exponential estimate for the tails of their probability distributions.

**Lemma 2.2 ([7], p. 68.)** *Let $\{X_i, i \geq 1\}$ be i.i.d. random variables with $\mathbf{P}[X_i = 1] = p$ and $\mathbf{P}[X_i = 0] = 1 - p$. Then for any $0 < p < a < 1$ and for any $N \geq 1$ we have*

$$\mathbf{P}\Big[\frac{1}{N}\sum_{i=1}^{N} X_i \geq a\Big] \leq \exp\{-NH(a,p)\}, \tag{2.2}$$

*where*

$$H(a,p) = a\log\frac{a}{p} + (1-a)\log\frac{1-a}{1-p} > 0. \tag{2.3}$$

Next large deviation result is an immediate consequence of Theorem 1.1, p. 748 of [5].

**Lemma 2.3** *Let $\{X_i, i \geq 1\}$ i.i.d. positive integer-valued random variables such that there are $C_1 > 0$ and $0 < \alpha < 1$ such that*

$$\mathbf{P}[X_i \geq n] \leq C_1 \exp\{-n^\alpha\}. \tag{2.4}$$

*Then there exist $a > 0,\ 0 < \beta < 1$ and $C_2 > 0$ such that for all $n$*

$$\mathbf{P}\Big[\sum_{i=1}^{n} X_i \geq an\Big] \leq C_2 \exp\{-n^\beta\}.$$

## 2.2 On simple random walks

The following results for $d-$dimensional SRW are found in [3] and [4]. Let $\mathsf{p}_n(x) = \mathbf{P}[S_n = x]$ and $\|x\|$ be the Euclidean norm. From Section 3 onwards we also work with the norm $\|x\|_1 = \|(x^{(1)}, \ldots, x^{(d)})\|_1 = \sum_{i=1}^{d} |x^{(i)}|$.

**Theorem 2.1 ([4], p. 14, 30.)**

$$\mathsf{p}_n(x) = \frac{2}{n^{d/2}}\Big(\frac{d}{2\pi}\Big)^{d/2} \exp\Big\{\frac{-d\|x\|^2}{2n}\Big\} + e_n(x), \tag{2.5}$$

*where $|e_n(x)| \leq Cn^{-(d+2)/2}$ for some $C$ and for all $a < d$*

$$\lim_{\|x\|\to\infty} \|x\|^a \sum_{n=1}^{\infty} |e_n(x)| = 0.$$

**Theorem 2.2 ([4], p. 29.)** *There is $C$ such that for all $n, t$*

$$\mathbf{P}[\sup_{0\leq i\leq n} \|S_i\| \geq tn^{1/2}] \leq Ce^{-t}. \tag{2.6}$$

Let

$$\mathsf{R}_n^{\mathsf{B}} = \{S_i^{\mathsf{B}} : 0 \le i \le n\} = \{y \in \mathbb{Z}^d : t(x,y) \le n \text{ for some } x \in \mathsf{B}\}$$

be the set of distinct sites visited by the family of SRWs, starting from the set of sites $\mathsf{B}$, up to time $n$. Some authors refer to $\mathsf{R}_n^0$ as the range of SRW. As usual, $|\mathsf{R}_n^{\mathsf{B}}|$ stands for the cardinality of $\mathsf{R}_n^{\mathsf{B}}$. A useful basic fact is that $|\mathsf{R}_n^{\mathsf{B}}| \le (n+1)|\mathsf{B}|$.

**Theorem 2.3 ([3], p. 333, 338.)**

- *If $d = 2$ then there is $a_2 > 0$ such that*

$$\lim_{n\to\infty} \frac{\mathbf{E}|\mathsf{R}_n^0|}{n/\log n} = a_2. \tag{2.7}$$

- *If $d \ge 3$ then there is $a_3 := a_3(d) > 0$ such that*

$$\lim_{n\to\infty} \frac{\mathbf{E}|\mathsf{R}_n^0|}{n} = a_3. \tag{2.8}$$

Let $G_n(x) := \sum_{j=0}^{n} \mathsf{p}_j(x)$ be the mean number of visits to site $x$ up to time $n$ and $G(x) = G_\infty(x)$. These are the well known *Green's functions*. Let $\mathsf{q}(n,x) = \mathbf{P}[t(0,x) \le n]$. From Theorem 2.1 we get the following result:

**Theorem 2.4**

- *If $d = 2, x \ne 0$ and $n \ge \|x\|^2$, then there exists $C_2 > 0$ such that*

$$\mathsf{q}(n,x) \ge \frac{C_2}{\log \|x\|}. \tag{2.9}$$

- *Suppose that $d \ge 3, x \ne 0$ and $n \ge \|x\|^2$. Then there exists $C_3 = C_3(d) > 0$ such that*

$$\mathsf{q}(n,x) \ge \frac{C_3}{\|x\|^{d-2}}. \tag{2.10}$$

*Proof.* Suppose without loss of generality that $\|x\|^2 \le n \le \|x\|^2 + 1$. Observe that

$$\begin{aligned} G_n(x) &= \sum_{j=0}^{n} \mathsf{p}_j(x) = \sum_{j=0}^{n}\sum_{k=0}^{j} \mathsf{p}_k(0)\mathbf{P}[t(0,x) = j-k] \\ &= \sum_{k=0}^{n} \mathsf{p}_k(0)\mathsf{q}(n-k,x) \le \mathsf{q}(n,x)G_n(0). \end{aligned}$$

So

$$\mathsf{q}(n,x) \geq \frac{G_n(x)}{G_n(0)} \geq \begin{cases} \dfrac{\sum_{j=\lfloor n/2 \rfloor}^{n} \mathsf{p}_j(x)}{\sum_{j=0}^{n} \mathsf{p}_j(0)}, & d=2, \\ (G(0))^{-1} \sum_{j=\lfloor n/2 \rfloor}^{n} \mathsf{p}_j(x), & d \geq 3. \end{cases}$$

Using (2.5), after some elementary computations we finish the proof. □

# 3 Subadditive ergodic theorem

The basic tools for proving shape theorems are the subadditive ergodic theorems. Next we state Liggett's subadditive ergodic theorem (cf., for example, [1]). In the sequence we apply it to the random variables $T(\cdot,\cdot)$.

**Theorem 3.1** *Suppose that* $\{Y(m,n)\}$ *is a collection of positive random variables indexed by integers satisfying* $0 \leq m < n$ *such that*

- $Y(0,n) \leq Y(0,m) + Y(m,n)$ *for all* $0 \leq m < n$ *(subadditivity);*
- *The joint distribution of* $\{Y(m+1, m+k+1), k \geq 1\}$ *is the same as that of* $\{Y(m, m+k), k \geq 1\}$ *for each* $m \geq 0$*;*
- *For each* $k \geq 1$ *the sequence of random variables* $\{Y(nk,(n+1)k), n \geq 1\}$ *is a stationary ergodic process;*
- $\mathbf{E}Y(0,1) < \infty$.

*Then*

$$\lim_{n\to\infty} \frac{Y(0,n)}{n} \to \gamma \qquad a.s.,$$

*where*

$$\gamma = \inf_{n \geq 0} \frac{\mathbf{E}Y(0,n)}{n}.$$

In order to verify the hypotheses of Liggett's subadditive ergodic theorem, for each fixed $x \in \mathbb{Z}^d$ let us consider $Y(m,n) = T(mx, nx)$.

First of all observe that the set of variables $\{T(x,y) : x, y \in \mathbb{Z}^d\}$ defined in Section 1 is *subadditive* in the sense that

$$T(x,z) \le T(x,y) + T(y,z) \text{ for all } x, y, z \in \mathbb{Z}^d. \tag{3.1}$$

Instead of proving this fact in a formal way, we prefer to give a verbal explanation. If site $z$ is reached before site $y$, there is nothing to prove. If that does not happen, observe that the process departuring from only site $y$ awakened (the one which gives the passage time $T(y,z)$) is coupled with the original process, for which one might have other particles awakened at time $T(x,y)$ besides that from $y$. Thus the remaining time to reach site $z$ for the original process can be at most $T(y,z)$. This takes care of the first hypothesis.

For a fixed $x \in \mathbb{Z}^d$ and $k \in \mathbb{N}$, the sequence $\{T((n-1)kx, nkx), n \ge 1\}$ is *stationary* by definition. *Ergodicity* holds because the sequence of random variables $\{T((n-1)kx, nkx), n \ge 1\}$ is strongly mixing. That can be checked easily because the events $\{T(n_1kx, (n_1+1)kx) = a\}$ and $\{T(n_2kx, (n_2+1)kx) = b\}$ are independent provided that $a + b < \|(n_1 - n_2)kx\|_1$.

It is simple to see that the fourth hypothesis holds when $d = 1$. To see that remember that for $\tau =$ *the first return to the origin of a SRW*, we can assure that $\mathbf{P}[\tau > t] \le Ct^{-1/2}$. Besides that, in a random time with exponential tail we will have at least three awakened particles jumping independently in the frog model. Combining these two facts we have that $\mathbf{E}T(0,1) < \infty$. So, for $d = 1$ one gets $T(0,n)/n \to \gamma$ a.s., and consequently we have the proof of the shape theorem with $\mathsf{A} = [-\gamma^{-1}, \gamma^{-1}]$ in dimension 1.

For higher dimensions we need a more powerful machinery in order to check the fourth hypothesis. This comes in the next result.

**Theorem 3.2** *For all $d \ge 2$ and $x_0 \in \mathbb{Z}^d$ there exist positive finite constants $C = C(x_0)$ and $\gamma$ such that*

$$\mathbf{P}[T(0,x_0) \ge n] \le C \exp\{-n^{\gamma}\}.$$

*Proof.* For technical reasons we treat the case $d = 2$ separately. Let $d = 2$. First pick $n \ge \|x_0\|^2$. Remember that

$$\mathsf{R}_n^0 = \{x \in \mathbb{Z}^d : t(0,x) \le n\}$$

is the range of the SRW of the first awakened particle, up to time $n$. By Theorem 2.3 we have that for all $k$ large enough it is true that $\mathbf{E}|\mathsf{R}_k^0| \ge$

$2C_1 k/\log k$. Since $|\mathsf{R}_k^0| \le k+1$, by using Lemma 2.1 we obtain

$$\mathbf{P}\Big[|\mathsf{R}_k^0| \ge \frac{C_1 k}{\log k}\Big] \ge \frac{C_2}{\log k}. \tag{3.2}$$

Divide the time interval $[0,n]$ into (roughly speaking) $n^{1/2}$ disjoint intervals of size $n^{1/2}$. Next we keep track of the displacement of the original particle over that time interval. For each subinterval of size $n^{1/2}$, the cardinality of the corresponding subrange does not depend on the cardinalities of other subranges. Therefore, for the event

$$A := \Big\{|\xi_n| \ge \frac{2C_1 n^{1/2}}{\log n}\Big\} \supset \Big\{|\mathsf{R}_n^0| \ge \frac{2C_1 n^{1/2}}{\log n}\Big\}$$

by (3.2) it holds that

$$\mathbf{P}[A] \ge 1 - \Big(1 - \frac{2C_2}{\log n}\Big)^{n^{1/2}}. \tag{3.3}$$

Let us now consider the event

$$B = B(n,\varepsilon) = \{\sup_{0\le i\le n} \|S_i^0\| < n^{1/2+\varepsilon}\}$$

where $0<\varepsilon<1$. Observe that by Theorem 2.2 there is $C_3$ such that

$$\mathbf{P}[B^c] \le C_3 \exp\{-n^{\varepsilon}\}. \tag{3.4}$$

Note that

$$\mathbf{P}[T(0,x_0) > n + 4n^{1+2\varepsilon}] \le \mathbf{P}[T(0,x_0) > n + 4n^{1+2\varepsilon} | A\cap B] + \mathbf{P}[A^c] + \mathbf{P}[B^c]. \tag{3.5}$$

Considering now all the particles awakened by the first particle, with the help of Theorem 2.4 one gets

$$\mathbf{P}[T(0,x_0) > n + 4n^{1+2\varepsilon} | A\cap B] \le \Big(1 - \frac{C_3}{\log n}\Big)^{2C_1 n^{1/2} (\log n)^{-1}} \tag{3.6}$$

for any fixed $0<\varepsilon<1$, so the result follows for $d=2$ from (3.3)–(3.6).

Now we treat higher dimension cases. For each dimension $d\ge 3$ fixed, the proof needs $\lfloor d/2\rfloor + 1$ steps. To do that in a general fashion, we separate

the proof in four parts named *first step, second step, general reasoning* and *denouement*. As the case $d = 3$ is simpler, we are able to finish its proof right after the *first step*. For $d = 4, 5$ we skip the part *general reasoning*, going directly to the part *denouement*. For higher dimensions all parts are needed.

*First step:*
Pick $n \geq \|x_0\|^2$. By Theorem 2.3 and Lemma 2.1 it follows that for some $r_1 > 0$

$$\mathbf{P}[|\mathsf{R}_k^0| \geq r_1 k] \geq C_1. \tag{3.7}$$

Divide the time interval $[0, n]$ into (roughly speaking) $n^\varepsilon$ disjoint intervals of size $n^{1-\varepsilon}$ for some $0 < \varepsilon < 1$ to be defined later. Next we keep track of the range of the original particle over that time interval. For each subinterval of size $n^{1-\varepsilon}$, the cardinality of the corresponding subrange does not depend on the cardinalities of other subranges. Let

$$A_1 := A_1(n, \varepsilon) := \{|\xi_n| \geq r_1 n^{1-\varepsilon}\} \supset \{|\mathsf{R}_n^0| \geq r_1 n^{1-\varepsilon}\};$$

by (3.7) it holds that

$$\mathbf{P}[A_1] \geq 1 - (1 - C_1)^{n^\varepsilon}. \tag{3.8}$$

Again consider the event

$$B = B(n, \varepsilon) = \{\sup_{0 \leq i \leq n} \|S_i^0\| < n^{1/2+\varepsilon}\}$$

and observe that by Theorem 2.2 there is $C_2$ such that

$$\mathbf{P}[B^c] \leq C_2 \exp\{-n^\varepsilon\}. \tag{3.9}$$

For $d = 3$ we are done, since by Theorem 2.4, analogously to (3.5)–(3.6) we have

$$\mathbf{P}[T(0, x_0) > n + 4n^{1+2\varepsilon} \mid A_1 \cap B] \leq \Big(1 - \frac{C_3}{n^{1/2+\varepsilon}}\Big)^{r_1 n^{1-\varepsilon}}.$$

By choosing $\varepsilon < 1/4$ and using (3.8)–(3.9), the result follows for $d = 3$.

*Second step:*
Denote

$$\mathsf{D}_{i,\varepsilon} := \{x \in \mathbb{Z}^d : \|x\| \leq i n^{1/2+\varepsilon}\}.$$

Suppose that events $A_1$ and $B$ happen. Suppose also that all the particles awakened by the original particle until time $n$ are allowed to start moving *exactly* at the moment $n$; clearly, such a procedure can only increase the hitting time of $x_0$ for the frog model. In this case, there are *at least* $r_1 n^{1-\varepsilon}$ active particles at time $n$ and all of them are inside the ball $\mathsf{D}_{1,\varepsilon}$. Choose $r_1 n^{1-\varepsilon}$ of these particles and call this set $G_1$. Split $G_1$ into $r_1 n^{\varepsilon}$ groups of size $n^{1-2\varepsilon}$. Call these groups $G_1^1, G_1^2, \dots, G_1^{r_1 n^{\varepsilon}}$. For each $y$ in the ring $\mathsf{D}_{2,\varepsilon} \setminus \mathsf{D}_{1,\varepsilon}$ let $\zeta_i^{(2)}(y)$ be the indicator function of the following event

$$\{\text{there exists } x \in G_1^1 \text{ such that } t(x,y) \le 9n^{1+2\varepsilon}\}.$$

By Theorem 2.4, and using the fact that for $x \in \mathsf{D}_{1,\varepsilon}$, $y \in \mathsf{D}_{2,\varepsilon} \setminus \mathsf{D}_{1,\varepsilon}$ the distance $\|x-y\|$ is less than or equal to $3n^{1/2+\varepsilon}$, we have

$$\begin{aligned}
\mathbf{E}(\zeta_i^{(2)}(y)|A_1 \cap B) &= \mathbf{P}[\zeta_i^{(2)}(y) = 1|A_1 \cap B] \\
&\ge 1 - \prod_{x \in G_1^i} (1 - \mathsf{q}(9n^{1+2\varepsilon}, y-x)) \\
&\ge 1 - \Big(1 - \frac{C_3}{3^{d-2} n^{(1/2+\varepsilon)(d-2)}}\Big)^{n^{1-2\varepsilon}} \\
&\ge \frac{C_4}{n^{d/2+d\varepsilon-2}} \qquad (3.10)
\end{aligned}$$

(here we used the fact that $d/2 + d\varepsilon > 2$ for $d \ge 4$). Let

$$\zeta_i^{(2)} = \sum_{y \in \mathsf{D}_{2,\varepsilon} \setminus \mathsf{D}_{1,\varepsilon}} \zeta_i^{(2)}(y).$$

Since $|\{x \in \mathbb{Z}^d : x \in \mathsf{D}_{2,\varepsilon} \setminus \mathsf{D}_{1,\varepsilon}\}| \ge C_5 n^{d/2+d\varepsilon}$, we have that there exists $r_2 > 0$ such that

$$\mathbf{E}(\zeta_i^{(2)}|A_1 \cap B) \ge 2r_2 n^2. \qquad (3.11)$$

Clearly, it is true that

$$\zeta_i^{(2)} \le n^{1-2\varepsilon} \times 4n^{1+2\varepsilon} = 4n^2. \qquad (3.12)$$

Therefore, using Lemma 2.1, we obtain from (3.11)–(3.12) that

$$\mathbf{P}[\zeta_i^{(2)} \ge r_2 n^2 |A_1 \cap B] \ge C_6 > 0. \qquad (3.13)$$

For $n_2 := n + 9n^{1+2\varepsilon}$ let

$$A_2 = A_2(n,\varepsilon) = \{|\xi_{n_2} \cap (\mathsf{D}_{2,\varepsilon} \setminus \mathsf{D}_{1,\varepsilon})| \geq r_2 n^2\}.$$

Repeating the above argument for the groups $G_1^2, \dots, G_1^{r_1 n^\varepsilon}$, we obtain from (3.13) that

$$\mathbf{P}[A_2^c | A_1 \cap B] \leq (1 - C_6)^{r_1 n^\varepsilon}. \tag{3.14}$$

For $d = 4$ or 5, one should go directly to *denouement*.

*General reasoning:*
Consider the sequence of times

$$n_1 := n, n_2 := n + 9n^{1+2\varepsilon}, \dots, n_k := n + n^{1+2\varepsilon} \sum_{j=2}^{k} (2j-1)^2,$$

$k \leq \lfloor d/2 \rfloor$. With that sequence, we define the events

$$A_k = \{|\xi_{n_k} \cap (\mathsf{D}_{k,\varepsilon} \setminus \mathsf{D}_{k-1,\varepsilon})| \geq r_k n^k\},$$

where the constants $r_k$, $k > 2$, will be defined later, and the random sets

$$\tilde{G}_k = \{x \in \mathsf{D}_{k,\varepsilon} \setminus \mathsf{D}_{k-1,\varepsilon} : T(0,x) < n_k\}.$$

for $k \in \{1, \dots, \lfloor d/2 \rfloor\}$. We claim that for $2 \leq k \leq \lfloor d/2 \rfloor - 1$

$$\mathbf{P}[A_{k+1} | A_k] \geq 1 - \exp\{-Cn^{2\varepsilon}\}. \tag{3.15}$$

To see why the claim is correct, pick $r_k n^k$ sites of $\tilde{G}_k$ (which are inside of the set $\mathsf{D}_{k,\varepsilon} \setminus \mathsf{D}_{k-1,\varepsilon}$) at time $n_k$ and divide them into $r_k n^{2\varepsilon}$ groups of size $n^{k-2\varepsilon}$. Name these sets $G_k^1, \dots, G_k^{r_k n^{2\varepsilon}}$. Name their union $G_k \subset \tilde{G}_k$. As before, we suppose that the particles that were originally at the sites of set $G_k$ begin to move only at time $n_k$. For each $y$ in the ring $\mathsf{D}_{k+1,\varepsilon} \setminus \mathsf{D}_{k,\varepsilon}$ let $\zeta_i^{(k+1)}(y)$ be the indicator function of the event

$$\{\text{there exists } x \in G_k^1 \text{ such that } t(x,y) \leq n_{k+1} - n_k\}.$$

Note that the quantities $n_k$ were defined in such a way that if $x \in \mathsf{D}_{k,\varepsilon}$, $y \in \mathsf{D}_{k+1,\varepsilon}$, then $\|x - y\| \leq n_{k+1} - n_k$. So, by Theorem 2.4, analogously to (3.10), we have

$$\mathbf{E}(\zeta_i^{(k+1)}(y) | A_k) \geq \frac{C_7}{n^{d/2+d\varepsilon-(k+1)}}$$

(note that $d/2 + d\varepsilon > k+1$ for $k \le \lfloor d/2 \rfloor - 1$). Let

$$\zeta_i^{(k+1)} = \sum_{y \in \mathsf{D}_{k+1} \setminus \mathsf{D}_k} \zeta_i^{k+1}(y).$$

Analogously, it follows that there exists $r_{k+1} > 0$ such that

$$\mathbf{E}(\zeta_i^{(k+1)} | A_k) \ge 2 r_{k+1} n^{k+1}$$

and, clearly,

$$\zeta_i^{(k+1)} \le n^{k-2\varepsilon} \times (2k+1)^2 n^{1+2\varepsilon} = (2k+1)^2 n^{k+1}.$$

So, by Lemma 2.1, there is $C_8$ such that

$$\mathbf{P}[\zeta_i^{(k+1)} \ge r_{k+1} n^{k+1} | A_k] \ge C_8 > 0,$$

so, considering now all the $r_k n^{2\varepsilon}$ groups, one gets

$$\mathbf{P}[A_{k+1}^c | A_k] \le (1 - C_8)^{r_k n^{2\varepsilon}},$$

which by its turn is equivalent to (3.15). Now by (3.8)–(3.9), (3.14)–(3.15) and using the following inequality

$$\mathbf{P}[A_{\lfloor d/2 \rfloor}] \ge \mathbf{P}[A_{\lfloor d/2 \rfloor} | A_{\lfloor d/2 \rfloor - 1}] \cdots \mathbf{P}[A_3 | A_2] \mathbf{P}[A_2 | A_1 \cap B] \mathbf{P}[A_1 \cap B]$$

it follows that

$$\mathbf{P}[A_{\lfloor d/2 \rfloor}] \ge 1 - C \exp\{-n^{\gamma_1}\} \tag{3.16}$$

for some $\gamma_1 > 0$.

*Denouement:*
Denote

$$I = \sum_{i=2}^{\lfloor d/2 \rfloor} (2i-1)^2.$$

The idea is to consider the particles in $\tilde{G}_{\lfloor d/2 \rfloor}$ at the moment $n_{\lfloor d/2 \rfloor} = n + I n^{1+2\varepsilon}$ and wait until the moment $n_{\lfloor d/2 \rfloor} + d^2 n^{1+2\varepsilon}$ in order to have a overwhelming probability for them to reach site $x_0$.

Let

$$H := \{\text{no hitting at } x_0 \text{ over time interval } (n_{\lfloor d/2 \rfloor}, n_{\lfloor d/2 \rfloor} + d^2 n^{1+2\varepsilon}]\}.$$

The number of particles in $\tilde{G}_{\lfloor d/2 \rfloor}$ is at least $r_{\lfloor d/2 \rfloor} n^{\lfloor d/2 \rfloor}$ and they are all at the distance of order $n^{1/2+\varepsilon}$ from $x_0$, so by using Theorem 2.4, we obtain

$$\begin{aligned} \mathbf{P}[T(0, x_0) > n + (I + d^2)n^{1+2\varepsilon} | A_{\lfloor d/2 \rfloor}] &\leq \mathbf{P}[H | A_{\lfloor d/2 \rfloor}] \\ &\leq \Big(1 - \frac{C_9}{n^{(1/2+\varepsilon)(d-2)}}\Big)^{r_{\lfloor d/2 \rfloor} n^{\lfloor d/2 \rfloor}}. \end{aligned}$$

Now, choosing $\varepsilon < \frac{1}{2(d-2)}$, and using the fact that

$$\begin{aligned} &\mathbf{P}[T(0, x_0) > n + (I + d^2)n^{1+2\varepsilon}] \\ &\quad \leq \mathbf{P}[T(0, x_0) > n + (I + d^2)n^{1+2\varepsilon} | A_{\lfloor d/2 \rfloor}] + \mathbf{P}[A^c_{\lfloor d/2 \rfloor}] \end{aligned}$$

together with (3.16), we are finished. □

*Remark.* The sub exponential estimate for the tail of the distribution of $T(0, x)$ also holds for $d = 1$. The proof is similar to what is done for $d = 2$ and therefore is omitted.

# 4 Asymptotic shape

In the previous section we proved that for all $x \in \mathbb{Z}^d$, the sequence $(T(nx, (n+1)x), n \geq 0)$ satisfies the hypotheses of Liggett's subadditive ergodic theorem. Therefore, defining $T(x) := T(0, x)$ for all $x \in \mathbb{Z}^d$, it holds that there exists $\mu(x) \geq 0$ such that

$$\frac{T(nx)}{n} \to \mu(x) \qquad \text{a.s., } n \to \infty. \tag{4.1}$$

From the fact $T(nx) \geq n\|x\|_1$ it follows that $\mu(x) \geq \|x\|_1$ for all $x \in \mathbb{Z}^d$.

**Lemma 4.1** *For all $x \in \mathbb{Z}^d$ there are constants $0 < \delta_0 < 1$, $C > 0$ and $0 < \gamma < 1$ such that*

$$\mathbf{P}\Big[T(x) \geq \frac{\|x\|_1}{\delta_0}\Big] \leq C \exp\{-\|x\|_1^\gamma\}.$$

*Proof.* Let $n := \|x\|_1$ and $0 = x_0, x_1, x_2, \dots, x_n = x$ be a path connecting the origin to site $x$ such that for all $i$, $\|x_i - x_{i-1}\|_1 = 1$. Let $Y_i := T(x_{i-1}, x_i)$. Due to the subadditivity, it is enough to proof that

$$\mathbf{P}\Big[\sum_{i=1}^{n} Y_i \geq \frac{\|x\|_1}{\delta_0}\Big] \leq C \exp\{-\|x\|_1^\gamma\}. \tag{4.2}$$

Let

$$B := \Big\{Y_i < \frac{\sqrt{n}}{2}, i = 1, \dots, n\Big\}.$$

Clearly, by Theorem 3.2 we have

$$\mathbf{P}[B] \geq 1 - C_1 n \exp\{-n^{\gamma'}\} \tag{4.3}$$

For some $\gamma' > 0$. For $i = 1, \dots, \lceil\sqrt{n}\rceil$ let

$$\sigma_i := \sum_{j=0}^{M_i} Y_{i+j\lceil\sqrt{n}\rceil},$$

where

$$M_i := \max\{j \in \mathbb{N} : i + j\lceil\sqrt{n}\rceil \leq n\}.$$

Observe that, if the event $B$ happens, then each $\sigma_i$ is as a sum of independent identically distributed random variables, since in this situation the variables $\{Y_{i+j\lceil\sqrt{n}\rceil} : j = 1, \dots, M_i\}$ depend on disjoint sets of random walks.

We point out that we cannot guarantee the existence of the moment generating function of $Y_i$. All we have is a sub exponential estimate as in (2.4) (see Theorem 3.2). Lemma 2.3 takes care of the situation and allows us to obtain (for $\delta_0 = 1/a$)

$$\begin{aligned}
\mathbf{P}\Big[\sum_{i=1}^{n} Y_i > \frac{n}{\delta_0}\Big| B\Big] &\leq \mathbf{P}\Big[\Big\{\sigma_1 \leq \frac{M_1}{\delta_0}, \dots, \sigma_{\lceil\sqrt{n}\rceil} \leq \frac{M_{\lceil\sqrt{n}\rceil}}{\delta_0}\Big\}^c \Big| B\Big] \\
&\leq \sum_{i=1}^{\lceil\sqrt{n}\rceil} \mathbf{P}\Big[\sigma_i \geq \frac{M_i}{\delta_0}\Big| B\Big] \\
&\leq C_2 \sum_{i=1}^{\lceil\sqrt{n}\rceil} \exp\{-M_i^\beta\}. \qquad (4.4)
\end{aligned}$$

Note that if $n$ is large then for all $i \leq \lceil \sqrt{n} \rceil$ it holds that $M_i = \mathcal{O}(\sqrt{n})$. The result follows from (4.3) and (4.4). $\square$

Let us extend the definition of $T(x, y)$ to the whole $\mathbb{R}^d \times \mathbb{R}^d$ by defining

$$\bar{T}(x, y) = \min\{n : y \in \bar{\xi}_n^{x_0}\},$$

where $x_0 \in \mathbb{Z}^d$ is such that $x \in (-1/2, 1/2]^d + x_0$. Note that the subadditive property holds for $\bar{T}(x, y)$ as well. The next goal is to show that $\mu$ can be extended to $\mathbb{R}^d$ in such a way that (4.1) holds for all $x \in \mathbb{R}^d$. As we did before, let us consider $\bar{T}(x) = \bar{T}(0, x)$.

**Lemma 4.2** *For all $x \in \mathbb{Q}^d$*

$$\frac{\bar{T}(nx)}{n} \to \frac{\mu(mx)}{m},$$

*where $m$ is the smallest positive integer such that $mx \in \mathbb{Z}^d$.*

*Proof.* Let $n = km + r$, where $k, r \in \mathbb{N}$ and $0 \leq r < m$. Since $\bar{T}(nx) \leq T(kmx) + \bar{T}(rx)$, it is true that

$$\limsup_{n\to\infty} \frac{\bar{T}(nx)}{n} \leq \frac{\mu(mx)}{m}. \tag{4.5}$$

Analogously, writing $n = (k+1)m - l$ one gets $T((k+1)mx) - \bar{T}(lx) \leq \bar{T}(nx)$, which implies that

$$\liminf_{n\to\infty} \frac{\bar{T}(nx)}{n} \geq \frac{\mu(mx)}{m}. \tag{4.6}$$

Combining (4.5) and (4.6), we finish the proof of Lemma 4.2. $\square$

By standard methods one can prove that $\mu$ is uniformly continuous in $\mathbb{Q}^d$, and therefore can be continuously extended to $\mathbb{R}^d$ in such a way that

$$\lim_{n\to\infty} \frac{\bar{T}(nx)}{n} = \mu(x). \tag{4.7}$$

So, it follows that $\mu$ is a norm in $\mathbb{R}^d$.

For $y \in \mathbb{R}^d$ and $a > 0$ denote $D(y, a) = \{x \in \mathbb{R}^d : \|x - y\|_1 \leq a\}$.

**Lemma 4.3** *There exist* $0 < \delta < 1$*,* $C > 0$*,* $\gamma_0 > 0$ *such that*

$$\mathbf{P}[D(0, n\delta) \subset \bar{\xi}_n] \geq 1 - C \exp\{-n^{\gamma_0}\}$$

*for all* $n$ *large enough.*

*Proof.* By Lemma 4.1, if $x$ is such that $\|x\|_1 = n$ then we have

$$\mathbf{P}\Big[T(x) \geq \frac{n}{\delta_0}\Big] \leq C \exp\{-n^{\gamma}\}.$$

Now, let $y \in \mathbb{Z}^d$ be such that $\|y\|_1 < n$. Then there exists $x \in \mathbb{Z}^d$ such that $\|x\|_1 = n$ and $\|x - y\|_1 = n$. As $T(y) \leq T(x) + T(x, y)$ and by Lemma 4.1 we have

$$\begin{aligned} \mathbf{P}\Big[T(y) \geq \frac{2n}{\delta_0}\Big] &\leq \mathbf{P}\Big[T(x) + T(x, y) \geq \frac{2n}{\delta_0}\Big] \\ &\leq \mathbf{P}\Big[T(x) \geq \frac{n}{\delta_0}\Big] + \mathbf{P}\Big[T(x - y) \geq \frac{n}{\delta_0}\Big] \\ &\leq 2C \exp\{-n^{\gamma}\}. \end{aligned}$$

So,

$$\mathbf{P}\Big[\text{there exists } y \in D(0, n) \text{ such that } T(y) \geq \frac{2n}{\delta_0}\Big] \leq C_1 n^d \exp\{-n^{\gamma}\},$$

which finishes the proof of Lemma 4.3. □

Now we are able to finish the proof of the shape theorem for the frog model.

*Proof of Theorem 1.1.* Let $\mathsf{A} := \{x \in \mathbb{R}^d : \mu(x) \leq 1\}$. We first prove that

$$n(1 - \varepsilon)\mathsf{A} \subset \bar{\xi}_n \text{ for all } n \text{ large enough, almost surely.}$$

Since $\mathsf{A}$ is compact, there exist $\mathsf{F} := \{x_1, \dots, x_k\} \in \mathsf{A}$ such that $\mu(x_i) < 1$ for $i = 1, \dots, k$, and (with $\delta_0$ from Lemma 4.1)

$$\mathsf{A} \subset \bigcup_{i=1}^{k} D(x_i, \varepsilon\delta_0).$$

Note that (4.7) implies that $n\mathsf{F} \subset \bar{\xi}_n$ for all $n$ large enough almost surely.

By Lemma 4.3, we have almost surely that there exists $n_o$ such that for all $n \geq n_0$

$$nD((1-\varepsilon)x_i, \varepsilon\delta_0) \subset \bar{\xi}_{n\varepsilon}^{n(1-\varepsilon)x_i}, \text{ for all } i = 1, 2, \dots k$$

and this part of the proof is done.

Now we prove that

$$\bar{\xi}_n \subset n(1+\varepsilon)\mathsf{A} \text{ for all } n \text{ large enough almost surely.}$$

First choose $\mathsf{G} := \{y_1, \dots, y_k\} \subset 2\mathsf{A} \setminus \mathsf{A}$ such that

$$2\mathsf{A} \setminus \mathsf{A} \subset \bigcup_{i=1}^{k} D(y_i, \varepsilon(1+\varepsilon)^{-1}\delta_0).$$

Notice that $\mu(y_i) > 1$ for $i = 1, \dots, k$. Analogously, (4.7) implies that

$$n\mathsf{G} \cap \bar{\xi}_n = \emptyset \text{ for all } n \text{ large enough almost surely.} \tag{4.8}$$

Suppose that, with positive probability,

$$\bar{\xi}_n \not\subset n(1+\varepsilon)\mathsf{A} \text{ for infinitely many } n \in \mathbb{N} \tag{4.9}$$

Fixed a realization of the process such that (4.8) and (4.9) happens, choose $n_0$ so large that for $n > n_0$, $n\mathsf{G} \cap \bar{\xi}_n = \emptyset$ and choose $n > n_0$ such that $\bar{\xi}_n \not\subset n(1+\varepsilon)\mathsf{A}$.

Since $\bar{\xi}_n$ is connected and (4.9) holds, there is a site $x \in \bar{\xi}_n \cap (1+\varepsilon)n(2\mathsf{A} \setminus \mathsf{A})$. By Lemma 4.3, we can suppose that in the realization we are considering, $n_0$ is so large that for $n > n_0$, $\mathsf{D}(x, n\varepsilon\delta_0) \subset \bar{\xi}_{n(1+\varepsilon)}$.

Notice that, since $(1+\varepsilon)n(2\mathsf{A} \setminus \mathsf{A}) \subset \bigcup_{i=1}^{k} D((1+\varepsilon)ny_i, n\varepsilon\delta_0)$, we must have

$$n(1+\varepsilon)y_k \in \mathsf{D}(x, n\varepsilon\delta_0) \subset \bar{\xi}_{n(1+\varepsilon)}.$$

This contradicts (4.8), and, therefore, concludes the proof of the theorem. □

*Proof of Theorem 1.2.* To prove the theorem, it is enough to prove the following fact: for fixed $i, j$, there exists $\beta$ such that

$$\Theta_{ij}^{\beta} \subset \mathsf{A}_m. \tag{4.10}$$

Indeed, in this case (4.10) holds for all $i, j$ with the same $\beta$ by symmetry, hence $\Theta^\beta \subset \mathsf{A}_m$ by virtue of convexity of $\mathsf{A}_m$.

Now, the proof of (4.10) is just a straightforward adaptation of the proof of "flat edge" result of [2]. To keep the paper self-contained, let us outline the ideas of the proof. Suppose, without loss of generality, that $i = 1, j = 2$ and $m$ is even. We are going to prove that the frog model observed only on $\Lambda_{12} \cap \mathbb{Z}_+^d$ dominates the oriented percolation process in $\mathbb{Z}_+^2$ with parameter $\theta = 1 - (1 - (2d)^{-1})^{m/2}$. To show this, first suppose that initially for any $x$ all the particles in $x$ are labeled "$x{\rightarrow}$" or "$x{\uparrow}$" in such a way that $x$ contains exactly $m/2$ particles of each label. Define $e_1, e_2$ to be the first two coordinate vectors. The oriented percolation is then defined in the following way: For $x \in \Lambda_{12} \cap \mathbb{Z}_+^d$

- the bond from $x$ to $x + e_1$ is open if for the frog model at the moment next to that of activation of the site $x$ at least one particle labeled "$x{\rightarrow}$" goes to $x + e_1$;
- the bond from $x$ to $x+e_2$ is open if at that moment at least one particle labeled "$x{\uparrow}$" goes to $x + e_2$.

Clearly, the two above events are independent, and their probabilities are exactly $\theta$. So the frog model indeed dominates the oriented percolation in the following sense: if a site $x = (x^{(1)}, x^{(2)})$ (for the sake of brevity forget the zero coordinates from 3 to $d$) belongs to cluster of 0 in the oriented percolation, then in the frog model the corresponding site is awakened *exactly* at time $x^{(1)} + x^{(2)}$. Now it rests only to choose $m$ as large as necessary to make the oriented percolation supercritical ($\theta \to 1$ as $m \to \infty$) and use the result that (conditioned on the event that the cluster of 0 is infinite) the intersection of the cluster of 0 with the line $\{(x^{(1)}, x^{(2)}) : x^{(1)} + x^{(2)} = n\}$ grows linearly in $n$ (cf. [2]). $\square$

*Proof of Theorem 1.3.* Denote

$$\mathcal{D}_n = \{x \in \mathbb{Z}^d : \|x\|_1 \leq n\}.$$

Choose $\theta < 1$ such that $\delta < \theta d$. Start the process and wait until the moment $n^\theta$. As $\eta \geq 1$ a.s., by Lemma 4.3 there exists $C_1$ such that with probability at least $1 - \exp\{n^{\gamma_0}\}$ all the frogs in the ball of radius $C_1 n^\theta$ centered in 0 will be awake. Let $x_1, \dots, x_N$ be the sites belonging to that ball enumerated in some order. Let $\zeta_i$ be the indicator of the following event:

$$\{\text{in the initial configuration } x_i \text{ contains less than } (2d)^n \text{ particles}\}.$$

Clearly, the inequality (1.3) implies that $\mathbf{P}[\zeta_i = 1] \le 1 - n^{-\delta}$. As $N$ is of order $n^{\theta d}$, one can apply Lemma 2.2 with $p = 1 - n^{-\delta}$ and $a = 1 - n^{-\delta}/2$ to get that with probability at least $1 - \exp\{-C_2 n^{\theta d-\delta}\}$ at time $n^\theta$ one will have at least $C_3 n^{\theta d-\delta}$ activated sites $x_i$ with $\zeta_i = 0$. This in turn means that at time $n^\theta$ there are at least $C_3 n^{\theta d-\delta}(2d)^n$ active frogs in $\mathcal{D}_{n^\theta}$. Note the following simple fact: If $x$ contains at least $(2d)^n$ active particles and $\|x - y\|_1 \le n$, then until time $n$ with probability bounded away from 0 at least one of those particles will hit $y$. Using this fact, we get that with overwhelming probability all the frogs in the diamond $\mathcal{D}_{n-n^\theta}$ will be awake at time $n^\theta + n$, which completes the proof of Theorem 1.3. $\square$

# 5 Remarks about continuous time

A continuous-time version of the frog model can also be considered. The difference from the discrete-time frog model is of course that here the particles, after being activated, perform a continuous-time SRWs with jump rate 1. In the continuous-time context Theorem 1.1 also holds and its proof can be obtained just by following the steps of our proof for the discrete case. The only difficulty that arises is that for continuous time, it is not so evident that $\mu(x)$ (defined by (4.1)) is strictly positive for $x \neq 0$, i.e. we must rule out the possibility that the continuous-time frog model grows faster than linearly. To overcome that difficulty, note the following fact (compare with Lemma 9 on page 16 of Chapter 1 of [1]): there exist a positive number $C$ such that, being $\|x\|_1 \ge Cn$, $\mathbf{P}[T(0,x) < n]$ is exponentially small in $n$. This fact by its turn easily follows from a domination of the frog model by branching random walk.


# Acknowledgements

The authors have benefited on useful comments and suggestion from K. Ravishankar and L. Fontes and wish to thank them. E. Kira should be thanked for a careful reading of the first draft of this paper. F.P.M. is also indebted to R. Durrett and M. Bramson for discussions about a continuous-time version of the model.

# References


[1] R. Durrett (1988) *Lecture Notes on Particle Systems and Percolation.* Wadsworth.

[2] R. Durrett, T.M. Liggett (1981) The shape of the limit set in Richardson's growth model. *Ann. Probab.* **9** (2), 186–193.

[3] B.D. Hughes (1995) *Random Walks and Random Environments, vol. 1.* Clarendon press, Oxford.

[4] G.F. Lawler (1991) *Intersections of Random Walks.* Birkhäuser Boston.

[5] S.V. Nagaev (1979) Large deviations of sums of independent random variables. *Ann. Probab.* **7** (5), 745–789.

[6] S.Yu. Popov (2001) Frogs in random environment. *J. Statist. Phys.* **102** (1/2), 191–201.

[7] A. Shiryaev (1989) *Probability (2nd. ed.).* Springer, New York.